Stimulating students' creativity with tasks solved using precomputation and visualization

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Introduction

The usual approaches of solving problems:

- making observations,
- guessing some properties,
- deducing solutions,
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- guessing some properties,
- deducing solutions,
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An alternative:

- use a computer,
- experiment,
- visualize,
- precompute,
- generate data and analyze it.

Number game

Two-player game

- state: a pair (x, y) of positive integers $(x, y \leq 10^9)$
- move
  1. a player subtracts 1 from one of the numbers
  2. then divides both numbers by their GCD
- the player who gets $(1, 1)$ loses the game

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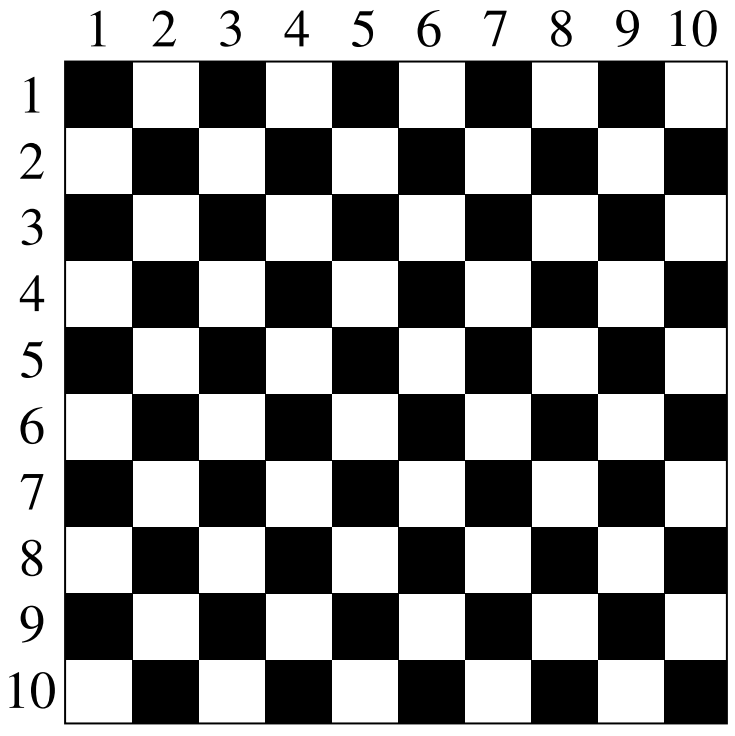
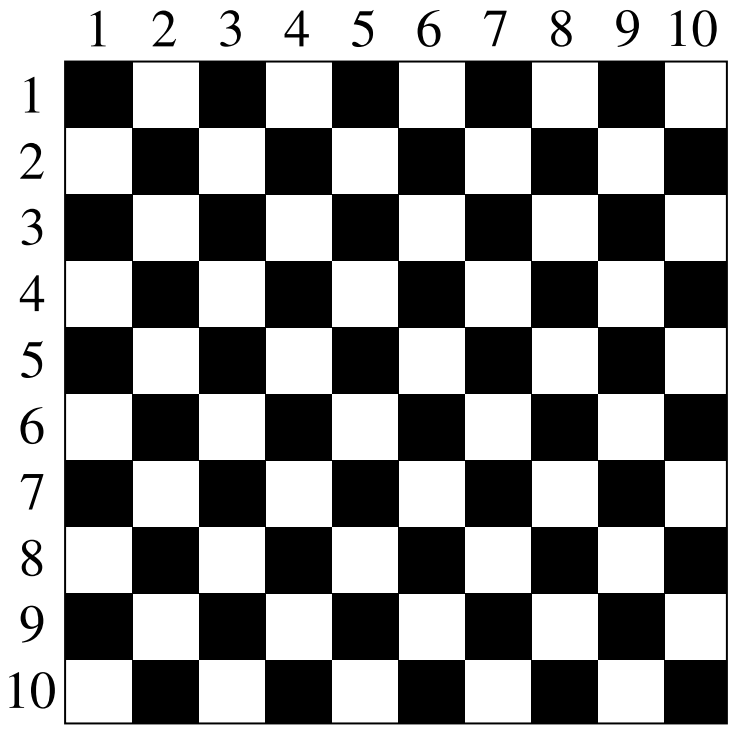
We can implement a simple DP algorithm (running in $O(n^2 \log n)$) to solve the problem for $i, j \leq n$.

Visualization



A cell $[i, j]$ is white if and only iff position (i, j) is winning.

Visualization



A cell $[i,j]$ is white if and only iff position (i,j) is winning.
We conjecture that (i,j) is losing if and only if i and j have the same parity.

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1. If both numbers have the same parity, then after the move they have different parity, so the position is losing.
2. If the parity is different, we subtract 1 from the even number. Hence, after the move both numbers are odd, so the opponent faces a losing position.

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Task (8th POI)
Compute the greatest antiprime which is lower than c ($c \leq 2 \cdot 10^9$).

Precomputation

Inefficient solution:

- We iterate over consecutive positive integers $\leq M$ and count their divisors.
- This requires $O(M\sqrt{M})$ time.
- Using Eratosthenes' sieve, one can improve the complexity to $O(M \log M)$ time.

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We discover that for $M = 10^6$ there are only 38 antiprimes.

Solution

A possible solution:

1. Implement the $O(M \log M)$ solution.
2. Run it for $M = 2 \cdot 10^9$.
3. Wait.
4. Hardcode the result into the program.

Possible improvements

The algorithm can be improved significantly.

- If $n = 2^{w_1} \cdot 3^{w_2} \cdot 5^{w_3} \cdot 7^{w_4} \cdot \ldots$ then n has $(w_1 + 1)(w_2 + 1)(w_3 + 1)(w_4 + 1) \ldots$ divisors.

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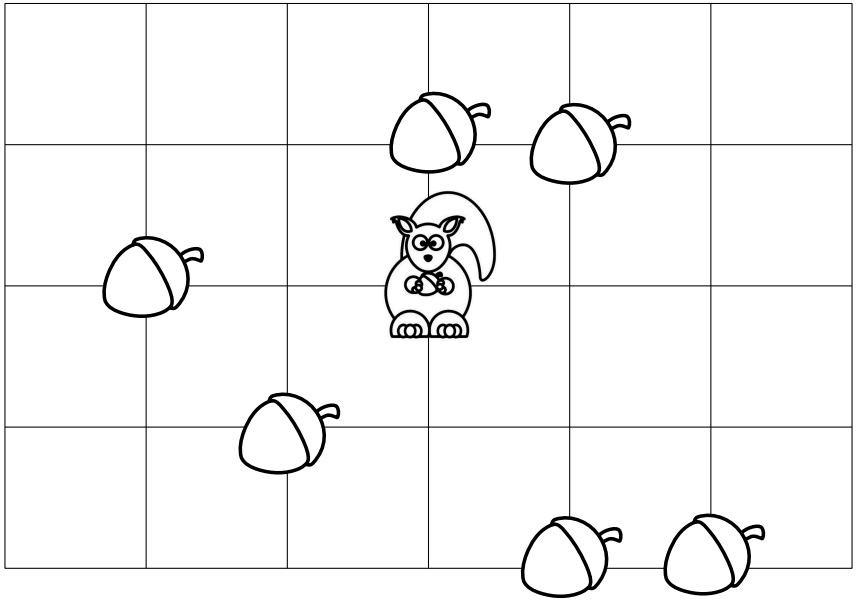
- We can only consider those numbers, where $w_1 \geq w_2 \geq w_3 \geq \ldots$.

Problem statement

Problem: A squirrel in a plane (ONTAK 2007)

Setting:

- infinite plane with $n \leq 7$ nuts,
- nuts placed at points (x_i, y_i) $-2 \leq x_i, y_i \leq 2$,
- a squirrel (initially at $(0, 0)$, facing north) collecting nuts.



Problem statement

Squirrel's moves:

- If it stands on a nut, then it
  - picks it up,
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- Otherwise, it
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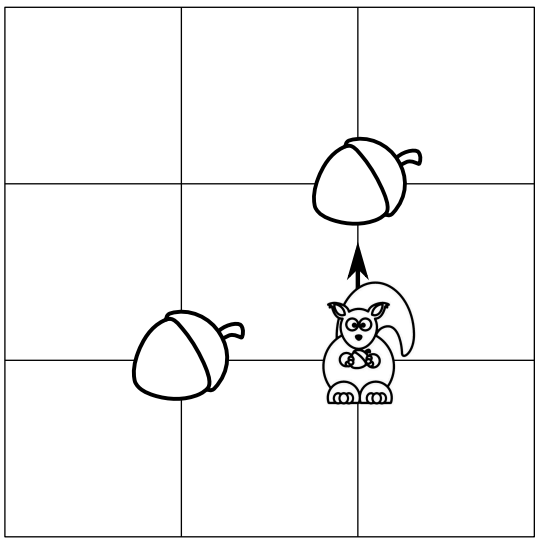

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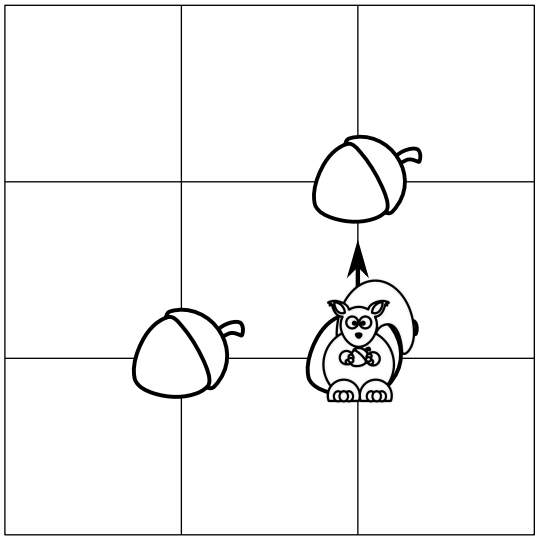

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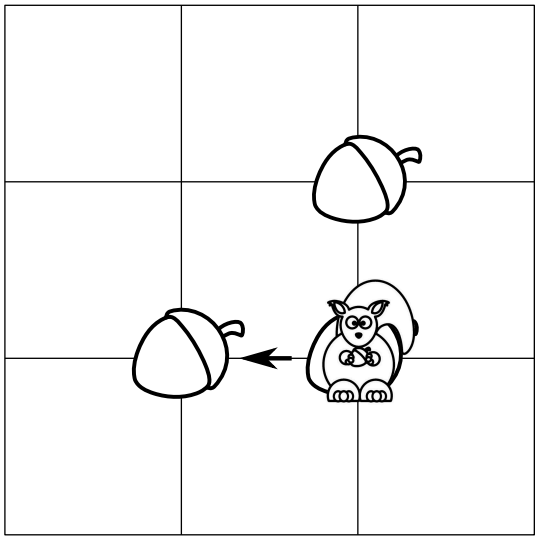

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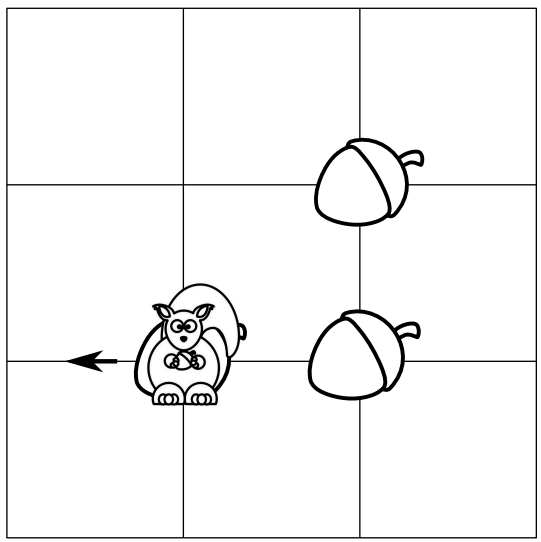

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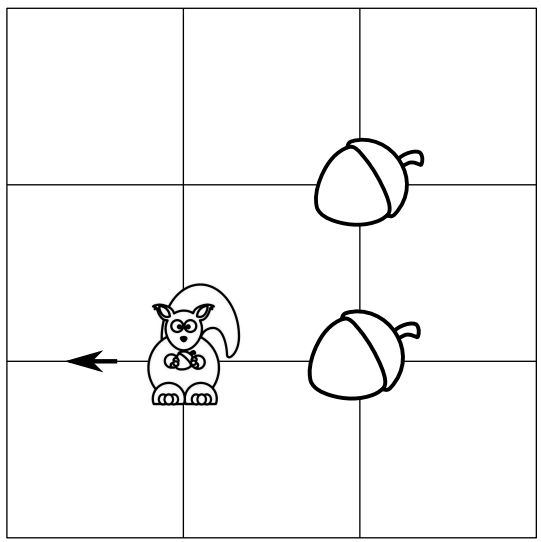

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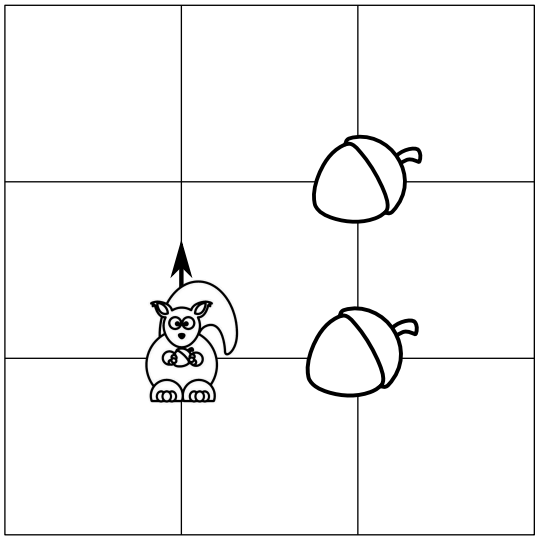

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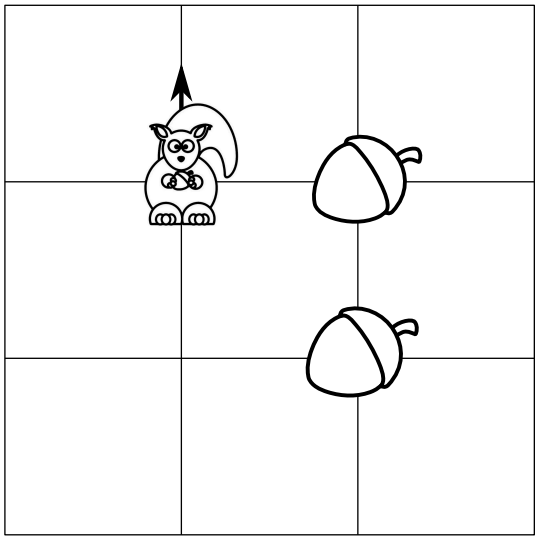

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Compute the number of nuts after $t \leq 10^9$ moves.

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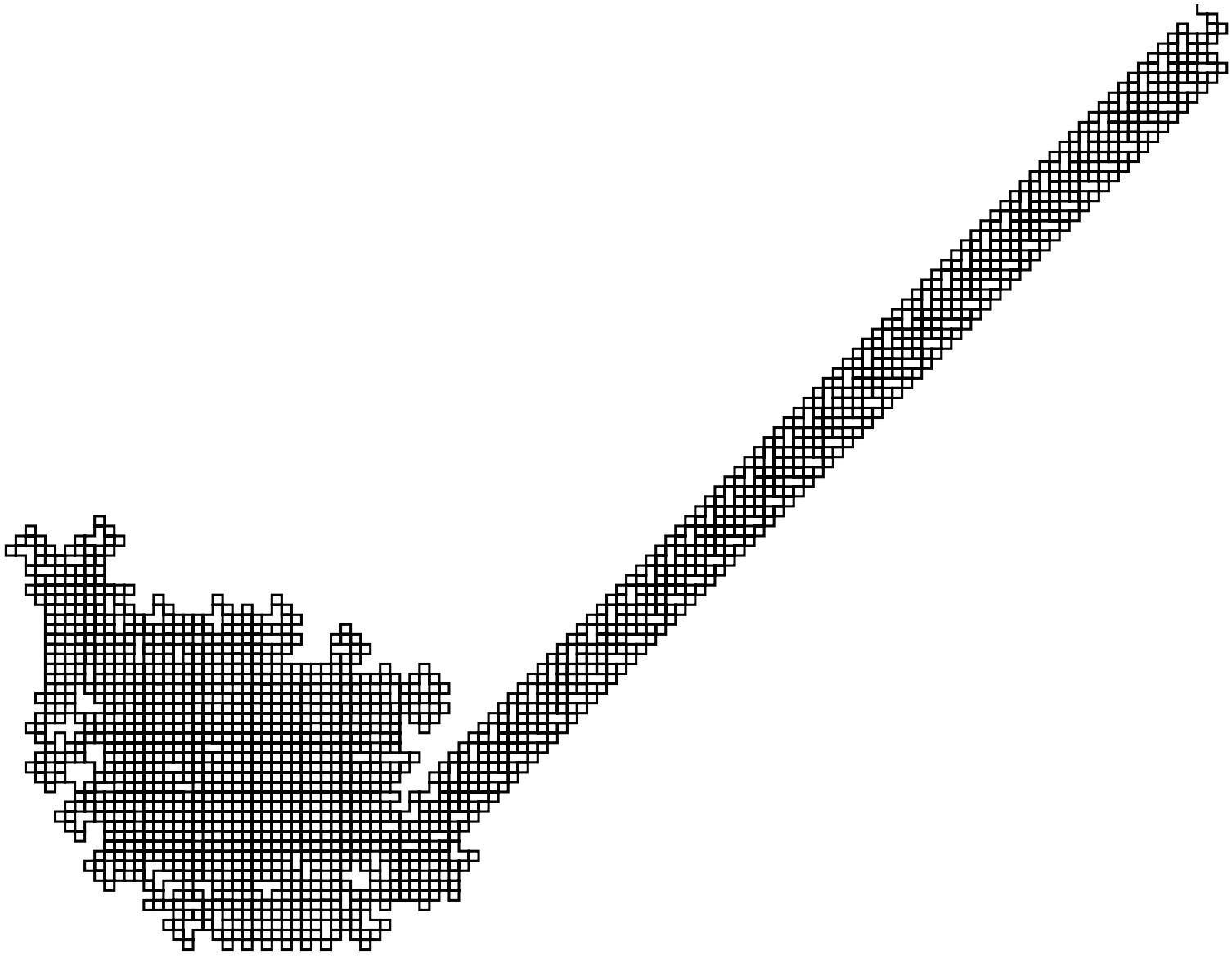
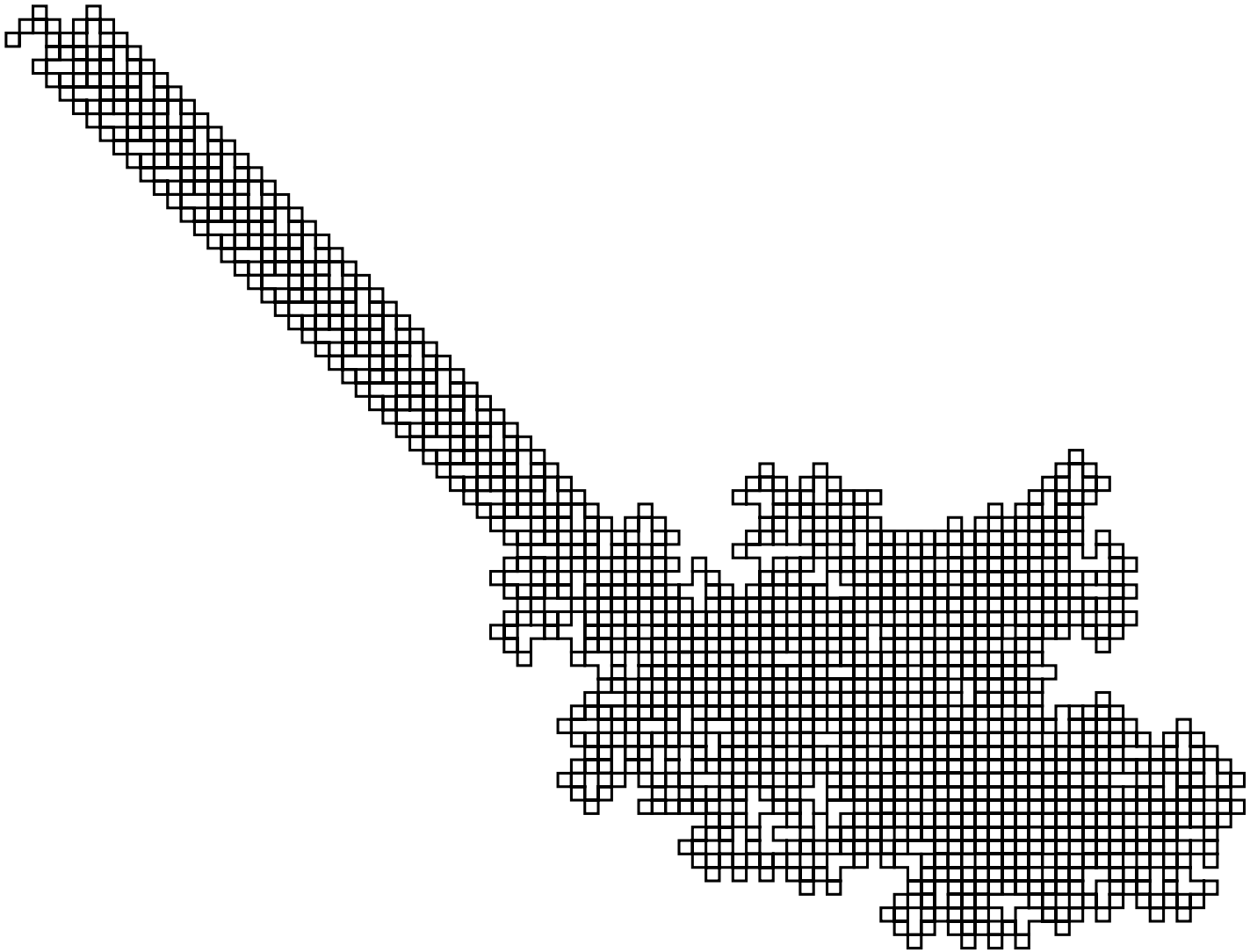
We simulate several thousand steps of the squirrel for different inputs.





Conclusions

Observations:

- The squirrel repeats a sequence of 104 moves,
- during which 12 nuts are put on the ground.
- The cycle starts close to the origin.

Solution

- We simulate the moves of the squirrel.
- The simulation stops after x moves if the squirrel
  - has reached the distance of 200 units from the origin and
  - $t - x$ is divisible by 104.
- We output the current number of nuts + $\frac{t-x}{104} \cdot 12$.

Correctness

How do we know if it is correct?

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Well, we can check all 726 206 initial configurations.

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$\left\{ {n \atop m} \right\}$ = number of partitions of $\{1, 2, \ldots, n\}$ into m nonempty sets.

E.g. $\left\{ {4 \atop 2} \right\} = 7$:

$$\{1,2,3\} \cup \{4\} \quad \{1,2,4\} \cup \{3\} \quad \{1,3,4\} \cup \{2\} \quad \{2,3,4\} \cup \{1\}$$

$$\{1,2\} \cup \{3,4\} \quad \{1,3\} \cup \{2,4\} \quad \{1,4\} \cup \{2,3\}$$

Problem description

Problem (ACM ICPC SWERC 2001)

Knowing that:

$$\left\{ {n \atop m} \right\} = \begin{cases} 0 & \text{if } n = m = 0 \\ 1 & \text{if } n = 0 \wedge m > 0 \text{ or } m = 0 \wedge n > 0 \\ m\left\{ {n-1 \atop m} \right\} + \left\{ {n-1 \atop m-1} \right\} & \text{if } n > 0 \wedge m > 0 \end{cases}$$

compute $\left\{ {n \atop m} \right\} \mod 2$ for $0 \leq m \leq n \leq 10^9$.

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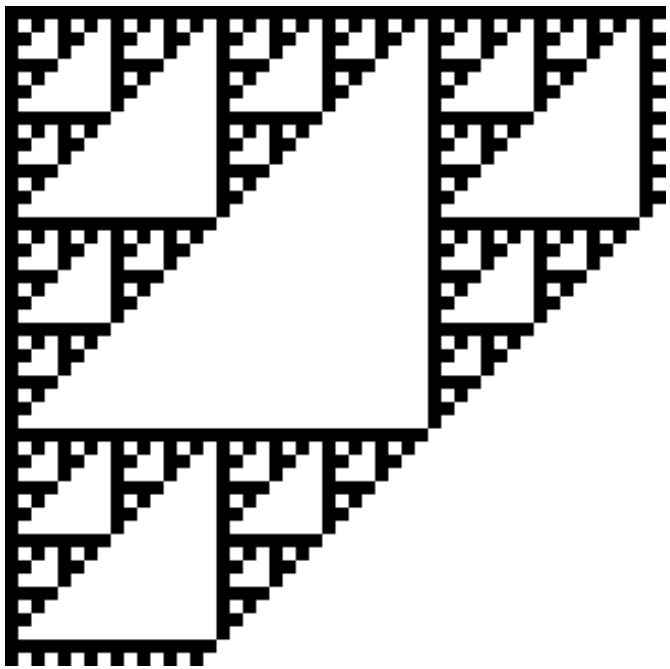
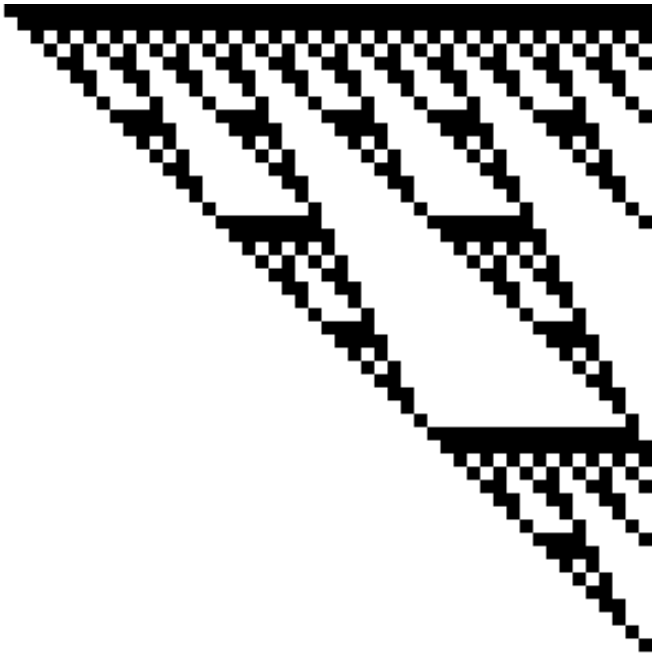
$$a[x][y] = \binom{x}{y} \mod 2$$



$$b[x][y] = 1 \iff x \& y = 0$$



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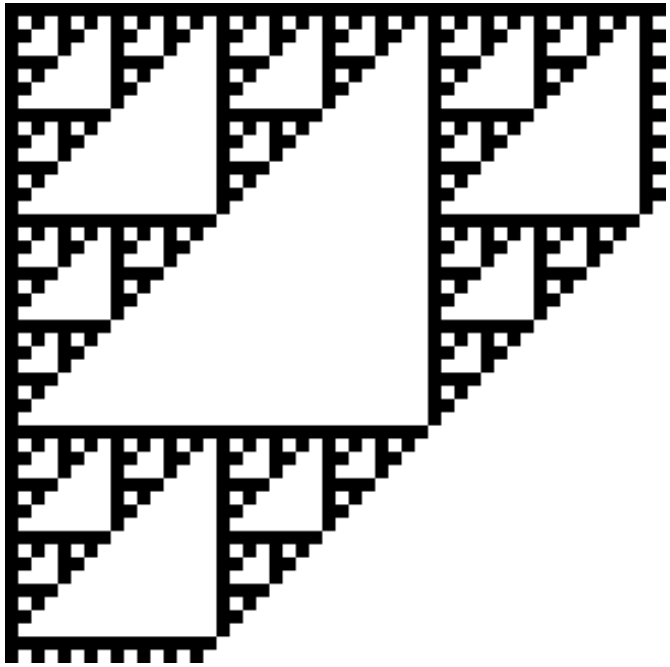

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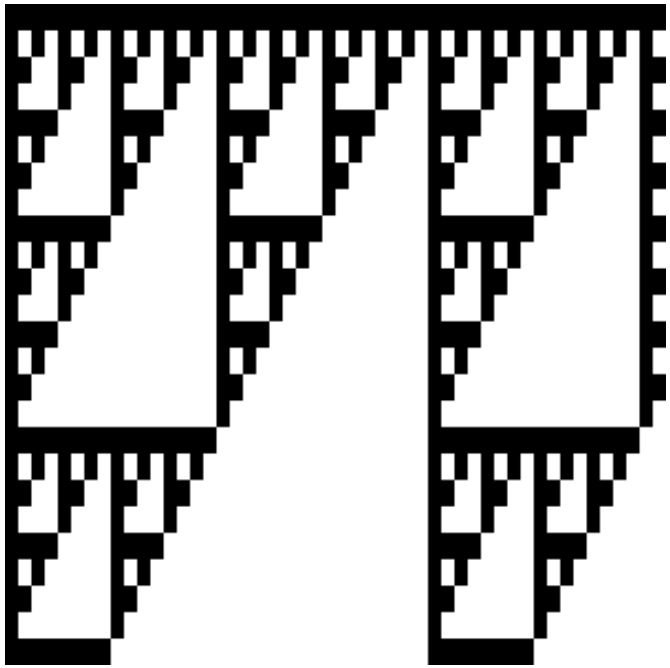

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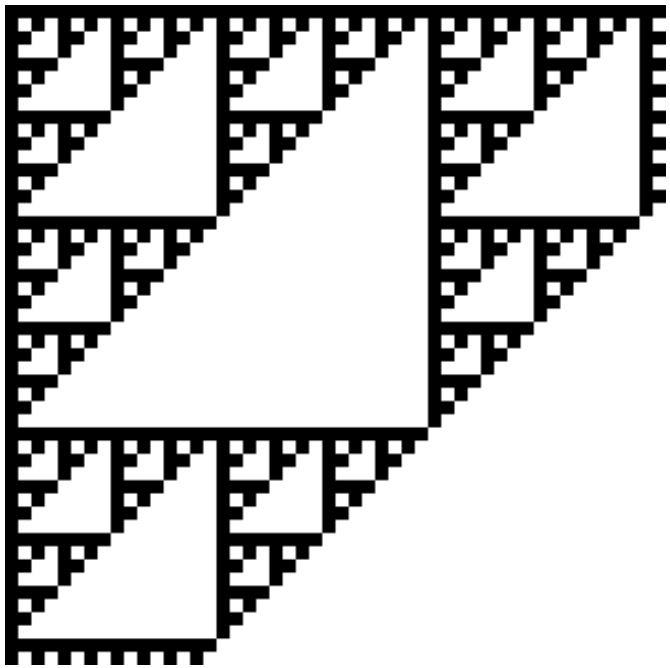

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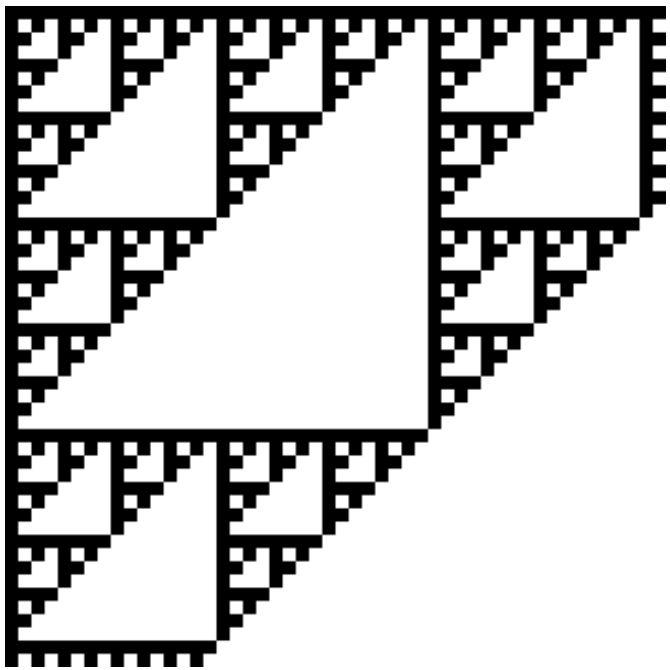

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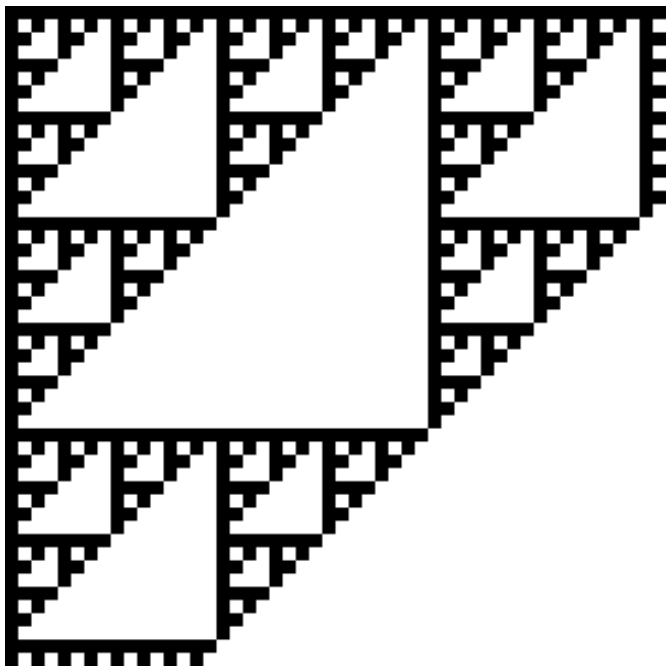

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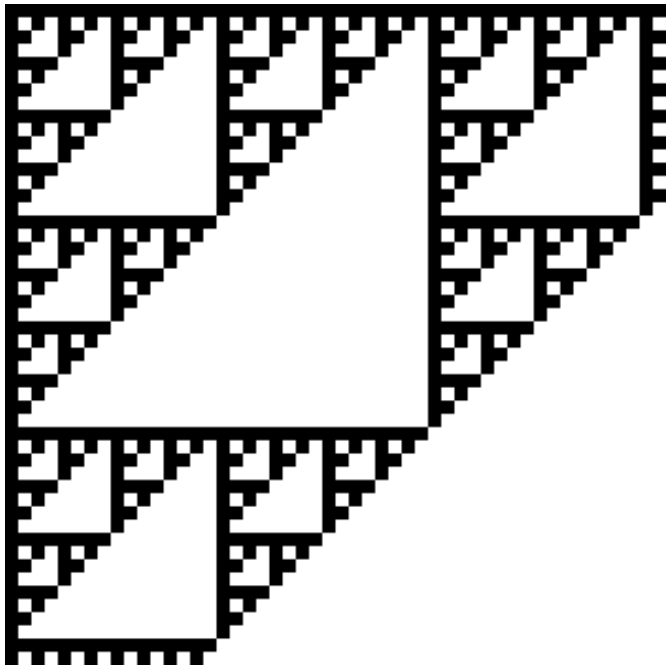

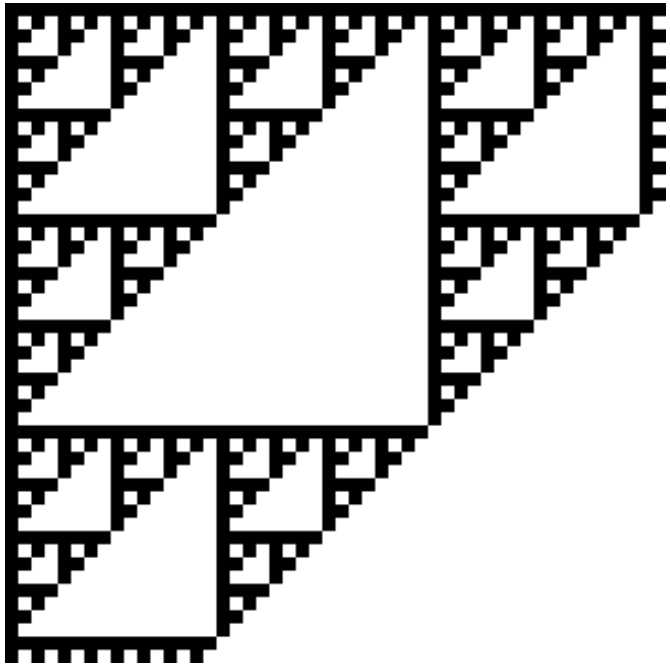

$a[x-y][\lfloor \frac{y-1}{2} \rfloor] = \binom{x}{y} \mod 2$

$b[x][y] = 1 \iff x \& y = 0$

We have $a[x][y] = b[x][y]$.

Conclusions

From

- $a[x-y][\lfloor \frac{y-1}{2} \rfloor] = \binom{x}{y} \mod 2$,
- $b[x][y] = 1 \iff x \& y = 0$,
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- $a[x-y][\lfloor \frac{y-1}{2} \rfloor] = \left\{ {x \atop y} \right\} \mod 2$,
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we get

$$\left\{ {x \atop y} \right\} \mod 2 = \left((x-y) \,\&\, \left\lfloor \frac{y-1}{2} \right\rfloor \right) == 0.$$

Summary

Solving tasks with a computer can:

- provide a deeper understanding for the student,
- develop engineering skills,
- stimulate creativity,
- teach students an alternative way of approaching problems,
- result in finding a better solution.